

Closing Thurs: HW 11.1(part 2), 12.1

Closing Tues: HW 12.4, 13.2

## Exam 2 is next Thursday!

Covers: 10.1-10.3: Analyzing a function

11.1,2: Deriv. of  $\ln(x)$  and  $e^x$

12.1,3,4: Integrals, finding C

13.2: Definite Integrals

Be able to answer any question about:

critical points, increasing/decreasing,  
local max/min, global max/min,  
concave up/down, inflection points,  
horizontal tangents

## 12.4: Antiderivatives and Applications

First, let's discuss how to find "C".

Entry Task: (from 12.4 HW)

Suppose  $MR(q) = 14 - 0.2q$ .

$$MC(q) = 30\sqrt{q+4}.$$

and fixed costs are  $FC = \$900$ .

Find the formulas for  $TR(q)$  and  $TC(q)$ .

$$\begin{aligned} TR(q) &= \int MR(q) dq \\ &= \int 14 - 0.2q \, dq \\ &= 14q - 0.2 \frac{1}{2} q^2 + C \end{aligned}$$

$$TR(q) = 14q - 0.1q^2 + C$$

Since  $TR(0) = 0 \leftarrow \text{ALWAYS TRUE!}$

$$14(0) - 0.1(0)^2 + C = 0 \Rightarrow C = 0$$

$$TR(q) = 14q - 0.1q^2 \quad \text{CHECK!!!}$$

$$TC(q) = \int 30(q+4)^{1/2} dq$$

$$TC(q) = 30 \frac{1}{3/2} (q+4)^{3/2} + C$$

$$TC(q) = 20(q+4)^{3/2} + C$$

$$TC(0) = 900 \leftarrow \text{GIVEN}$$

$$\text{So } 20(0+4)^{3/2} + C = 900$$

$$\Rightarrow 20 \cdot 8 + C = 900$$

$$\Rightarrow 160 + C = 900 \Rightarrow C = 740$$

$$TC(q) = 20(q+4)^{3/2} + 740 \quad \text{CHECK!!!}$$

**Note:** To integral  $MC(q)$  in the entry task, you had to guess and check a slightly varied version of the formula:

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C,$$

which is this

$$\int (x+a)^n dx = \frac{1}{n+1} (x+a)^{n+1} + C$$

You won't use this a lot, but are welcome to make a note of this more general version. This is the only slight variation from our four examples that you will see in homework.

**Please remember to always, always, check your antiderivatives (by differentiating)!!**

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

$$\int k dx = kx + C$$

$$\int \frac{1}{x} dx = \ln(x) + C$$

## HOW TO FIND "C"

**Step 1:** Integrate (don't forget +C)

**Step 2:** Plug in your initial condition.

(The given x and y values)

**Step 3:** Solve for C.

*Example:*

$$f'(x) = 8e^{4x} + \frac{1}{\sqrt{x}}, \text{ and } f(1) = 15$$

Find  $f(x)$ .

$$\begin{aligned} f(x) &= \int 8e^{4x} + x^{-1/2} dx \\ &= 8 \cdot \frac{1}{4} e^{4x} + \frac{1}{1/2} x^{1/2} + C \\ &= 2e^{4x} + 2\sqrt{x} + C \end{aligned}$$

$$\begin{aligned} f(1) = 15 &\Rightarrow 2e^4 + 2 + C = 15 \\ &\Rightarrow C = 13 - 2e^4 \end{aligned}$$

$$f(x) = 2e^{4x} + 2\sqrt{x} + 13 - 2e^4$$

CHECK!!!

*Example:*

$$f''(x) = -32, f'(0) = 0, f(0) = 100$$

Find  $f(x)$ .

$$f'(x) = \int -32 dx$$

$$f'(x) = -32x + C$$

$$\begin{aligned} f'(0) = 0 &\Rightarrow -32(0) + C = 0 \\ &\Rightarrow C = 0 \end{aligned}$$

$$\text{So } f'(x) = -32x$$

$$f(x) = \int -32x dx$$

$$= -16x^2 + C$$

$$\begin{aligned} f(0) = 100 &\Rightarrow -16(0)^2 + C = 100 \\ &C = 100 \end{aligned}$$

$$f(x) = -16x^2 + 100$$

## How to do all applications problems

### **Step 1:** What are you given?

Identify, label and use the connections to find all related functions.

### **Step 2:** What do you want?

What function is the question asking about? Write that function down first!!!  
That is your "original" function for this problem.

### **Step 3:** Translate. Solve.

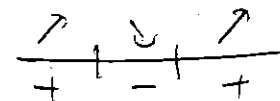
I gave two extensive handouts on how to translate and solve and you've done dozens in homework. You should know the methods well.

### **Step 4:** Present your answer.

Is the question asking for the *value* of the function (meaning output), or the input,  $x$ , where it occurs, or something else? Read carefully and appropriately interpret your work.

critical number, horizontal tangent  
 $f'(x) \stackrel{?}{=} 0$  solve

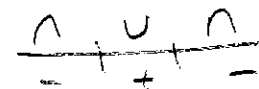
increasing/decreasing  
 $f'(x) = 0$  solve  
number line



local max/min  
 $f'(x) = 0$  solve  
number line on

2nd deriv. test  
 $f'(a) = 0$  &  $f''(a) > 0$  U  
 $f'(a) = 0$  &  $f''(a) < 0$  n

concave up/down, pts of inflection  
 $f''(x) = 0$  solve  
number line



global max/min  
 $f'(x) = 0$  solve  
plug critical pts  
& endpoints into original function

Example: (Old Exam Question – like HW)

Two vats have water coming in and out.

At time  $t$  hours, we define:

$A(t)$  = "gallons of water in Vat A"

$B(t)$  = "gallons of water in Vat B"

You are given

$$A'(t) = -3t^2 + 24t - 21 \quad \text{gal/hr}$$

$$B(t) = 3t - 9 \ln(t + 1) + 10 \quad \text{gallons}$$

The two vats contain the same amount of water at time  $t = 0$ .

- Find the formula for  $A(t)$ .
- Find and classify all critical numbers of  $A(t)$ .
- Find the global maximum of  $B(t)$  on the interval  $t = 0$  to  $t = 5$ .
- Give the longest interval over which the graph of  $A(t)$  is concave up.

$$\begin{aligned} (a) \quad A(t) &= \int -3t^2 + 24t - 21 \, dt \\ &= -3 \frac{1}{3} t^3 + 24 \frac{1}{2} t^2 - 21 t + C \\ A(t) &= -t^3 + 12t^2 - 21t + C \\ \text{AND } A(0) &= B(0) = 10 \quad \leftarrow \\ &\quad 3(0) - 9 \ln(1) + 10 = \\ \Rightarrow t(0)^3 + 12(0)^2 - 21(0) + C &= 10 \Rightarrow C = 10 \\ \boxed{A(t) = -t^3 + 12t^2 - 21t + 10} & \quad \text{CHECK!!} \end{aligned}$$

(b) WANT CRITICAL # of  $A(t)$ . ← original function

$$\text{FIND } A'(t) = -3t^2 + 24t - 21 \stackrel{?}{=} 0$$

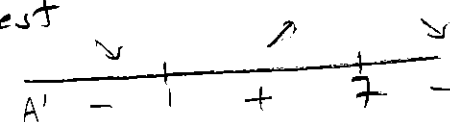
$$-3(t^2 - 8t + 7) \stackrel{?}{=} 0$$

$$m: -3(t-1)(t-7) = 0$$

$$t = 1 \quad \text{on} \quad t = 7$$

2 ways to classify

• 1<sup>st</sup> Deriv. Test



• 2<sup>nd</sup> Deriv. Test

$$A''(t) = -6t + 24$$

$$A''(1) = 18 > 0$$

U ← min

$$A''(7) = -18 < 0$$

∩ ← max

(c) WANT GLOBAL MAX  
OF  $B(t)$  ON  $t=0$  to  $t=5$   
↳ "original"

$$\text{Find } B'(t) = 3 - 9 \cdot \frac{1}{t+1} \cdot 1 \stackrel{?}{=} 0$$

$$3(t+1) - 9 = 0$$

$$3t + 3 - 9 = 0$$

$$3t = 6$$

$$t = 2$$

$$B(0) = 10$$

$$\begin{aligned} B(2) &= 3(2) - 9 \ln(2+1) + 10 \\ &= 16 - 9 \ln(3) \approx 6.1125 \end{aligned}$$

$$\begin{aligned} B(5) &= 3(5) - 9 \ln(5+1) + 10 \\ &= 25 - 9 \ln(6) \approx 8.8742 \end{aligned}$$

$$\boxed{\text{GLOBAL MAX} = 10}$$

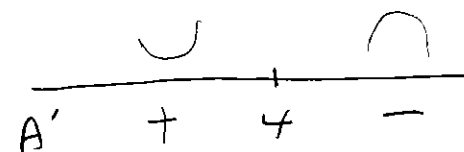
$$\boxed{\text{GLOBAL MIN} = 16 - 9 \ln(3)}$$

(d) WANT INTERVAL WHERE  
 $A(t)$  IS CONCAVE UP.  
↳ "original"

$$A''(t) = -6t + 24 \stackrel{?}{=} 0$$

$$-6t = -24$$

$$t = 4$$



$$\boxed{t < 4}$$

## Business Application Review:

We have ways to go between  
*all* our business functions!

### HW 12.4 Overview:

1. MR to TR.
  2. AC' to AC.
  3. MR to TR.
  4. MR/MC to TR/TC.
- etc....

Example (from HW):

$$AC'(q) = -\frac{4}{q^2} + \frac{1}{4}$$

$$AC(4) = 10$$

Find the formula for  $AC(q)$  and  $TC(q)$ . *what is FC?*

$$AC(q) = \int AC'(q) dq = \int -4q^{-2} + \frac{1}{4} dq = -4 \cdot \frac{1}{-1} q^{-1} + \frac{1}{4} q + C$$

$$AC(q) = \frac{4}{q} + \frac{1}{4} q + C$$

$$AC(4) = 10 \Rightarrow \frac{4}{4} + \frac{1}{4}(4) + C = 10 \Rightarrow 2 + C = 10 \Rightarrow \boxed{C=8}$$

$$AC(q) = \frac{4}{q} + \frac{1}{4} q + 8$$

$$TC(q) = q \cdot AC(q) = q \left( \frac{4}{q} + \frac{1}{4} q + 8 \right) = 4 + \frac{1}{4} q^2 + 8q$$

NOTE:  
 $TC(0) = 4 = FC$

Total Values and Marginal Values	Total Values and Average Values
$TR(x) = \int MR(x) dx$ $TR'(x) = MR(x)$	$AR(x) = \frac{TR(x)}{x} = \text{price}$ $TR(x) = xAR(x)$
$TC(x) = \int MC(x) dx$ $TC'(x) = MC(x)$	$AC(x) = \frac{TC(x)}{x}$ $TC(x) = xAC(x)$
$P(x) = \int MP(x) dx$ $P'(x) = MP(x)$	
Initial conditions: $TR(0) = 0, \quad TC(0) = FC$	
$P(x) = TR(x) - TC(x)$	

1. (16 points) Let  $f(x) = 4x^3 - 78x^2 + 432x$ .

- (a) Find all critical numbers of  $f(x)$  and use the Second Derivative Test to determine whether each gives a local maximum or a local minimum value of  $f(x)$ .

**STEP 1**  $f'(x) = 12x^2 - 156x + 432 \stackrel{?}{=} 0$  FACTOR OUT 12  
 $12(x^2 - 13x + 36) \stackrel{?}{=} 0$   
 $12(x - 4)(x - 9) \stackrel{?}{=} 0 \Rightarrow \boxed{x=4 \text{ or } x=9}$  OR USE QUAD. FORMULA

**STEP 2**  $f''(x) = 24x - 156$   
 At  $x=4$ ,  $f''(4) = 24(4) - 156 = -60$  :  $\left. \begin{array}{l} f'(4)=0 \\ f''(4)<0 \end{array} \right\}$  **LOCAL MAX**  
 At  $x=9$ ,  $f''(9) = 24(9) - 156 = 60$  :  $\left. \begin{array}{l} f'(9)=0 \\ f''(9)>0 \end{array} \right\}$  **LOCAL MIN**  
 ANSWER:  $x = \underline{4}$  gives a local **MAX**  
 ANSWER:  $x = \underline{9}$  gives a local **MIN**

- (b) Define a new function  $D(x)$  by  $D(x) = \frac{f(x)}{x}$ . Find the value of  $x$  at which  $D(x)$  reaches its smallest value. (Your work should include an explanation of how you know  $D(x)$  is smallest there.)

**STEP 1**  $D(x) = 4x^2 - 78x + 432 \Rightarrow D'(x) = 8x - 78 \stackrel{?}{=} 0$   
 $8x = 78 \Rightarrow x = \frac{78}{8} = 9.75$

**STEP 2**  $D \searrow \quad \nearrow$   
 $D' \quad - \quad 9.75 \quad +$   
 For  $x < 9.75$ ,  $D'(x) = -78 < 0$   
 For  $x > 9.75$ ,  $D'(x) = 8(10) - 78 = 2 > 0$   
 } THUS, AT  $x = 9.75$   
 $D(x)$  MUST REACH ITS SMALLEST VALUE

ANSWER:  $x = \underline{9.75}$

- (c) Define a new function  $S(x)$  by  $S(x) = \frac{D(x)}{x}$ . Find all positive critical numbers of  $S(x)$ .

**STEP 1**  $S(x) = 4x - 78 + \frac{432}{x} = 4x - 78 + 432x^{-1}$   
 $\Rightarrow S'(x) = 4 - 432x^{-2} = 4 - \frac{432}{x^2} \stackrel{?}{=} 0$   
 $\Rightarrow 4x^2 - 432 = 0 \Rightarrow 4x^2 = 432 \Rightarrow x^2 = 108$   
 $x = \pm \sqrt{108}$

ANSWER:  $x = \underline{\pm \sqrt{108} \approx 10.392}$



2. (19 points) You sell Gizmos. Your total revenue and total cost are given by the functions  $TR(q) = -2q^2 + 199.1q$  and  $TC(q) = 0.01q^3 - 2.405q^2 + 200q + 20$ , where  $q$  is in thousands of Gizmos and  $TR$  and  $TC$  are both in thousands of dollars.

(a) Find the largest interval on which  $MR(q)$  is positive.

STEP 1  $MR(q) = -4q + 199.1 = 0 \Rightarrow 199.1 = 4q \Rightarrow q = \frac{199.1}{4} = 49.775$

STEP 2  $\begin{array}{c} \nearrow \\ \text{MR} \end{array} \begin{array}{c} + \\ 49.775 \end{array} \begin{array}{c} \searrow \\ - \end{array}$

ANSWER: from  $q = 0$  to  $q = 49.775$  thousand Gizmos

(b) Is  $TC(q)$  concave up or concave down at  $q = 100$ ?

STEP 1  $TC'(q) = 0.03q^2 - 4.81q + 200$   
 $TC''(q) = 0.06q - 4.81$

STEP 2  $TC''(100) = 0.06(100) - 4.81 = 6 - 4.81 = 1.19 > 0$

ANSWER: (circle one) concave up concave down

(c) Recall that  $FC = TC(0)$ ,  $TC(q) = VC(q) + FC$ , and  $AVC(q) = \frac{VC(q)}{q}$ . Find all critical numbers of  $AVC(q)$ .

STEP 1  $AVC(q) = \frac{0.01q^3 - 2.405q^2 + 200q}{q} = 0.01q^2 - 2.405q + 200$   
 $\Rightarrow AVC'(q) = 0.02q - 2.405 = 0$   
 $q = \frac{2.405}{0.02} = 120.25$

ANSWER: (list all)  $q = 120.25$  thousand Gizmos

(d) Let  $P(q)$  denote the profit (in thousands of dollars) at  $q$  thousand Gizmos. The critical numbers of  $P(q)$  are  $q = 1.16$  and  $q = 25.84$  thousand Gizmos. Determine whether each critical number gives a local minimum of  $P(q)$ , a local maximum of  $P(q)$ , or neither.

STEP 1  $P'(q) = (-4q + 199.1) - (0.03q^2 - 4.81q + 200)$

$\Rightarrow P'(q) = -0.03q^2 + 0.81q - 0.9$

$P''(q) = -0.06q + 0.81$

$q = 1.16 \Rightarrow P''(1.16) = -0.06(1.16) + 0.81 = 0.7404 > 0$

$q = 25.84 \Rightarrow P''(25.84) = -0.06(25.84) + 0.81 = -0.7404 < 0$

ANSWER:  $q = 1.16$  gives a (circle one) local min local max neither

$q = 25.84$  gives a (circle one) local min local max neither

4. (13 pts) Your Total Cost (in hundreds of dollars) and Demand Curve (in dollars) vs. the quantity  $q$  in hundreds of Items sold is given by the function:

$$TC(q) = \frac{q^3}{12} - \frac{q^2}{2} + \frac{3}{4}q + 10 \quad \text{and} \quad p = h(q) = 24 - 8\sqrt{q}.$$

- (a) (7 pts) Write the formula for **Total Revenue**,  $TR$ , and give the **prices** that correspond to the global maximum and global minimum value of **Total Revenue** over the interval  $q = 2$  to  $q = 6$  hundred Items.

**STEP 1**  $TR(q) = 24q - 8q^{3/2}$   
 $TR'(q) = 24 - 8 \cdot \frac{3}{2} q^{1/2} = 24 - 12q^{1/2} \stackrel{?}{=} 0$   
 $q^{1/2} = 2 \Rightarrow q = 4$

**STEP 2**  $TR(2) = 24(2) - 8(2)^{3/2} \approx 25.3726 \leftarrow \text{min}$   
 $TR(4) = 24(4) - 8(4)^{3/2} = 32 \leftarrow \text{max}$   
 $TR(6) = 24(6) - 8(6)^{3/2} = 26.4245$

**STEP 3** Corresponding Price?  
 $p = h(2) = 24 - 8\sqrt{2} = 12.687$   
 $p = h(4) = 24 - 8\sqrt{4} = 8$

ANSWER: **PRICE** for the global minimum value = 12.69 dollars

**PRICE** for the global maximum value = 8 dollars

- (b) (6 pts) Find **all** critical numbers of **Total Cost**,  $TC$ . Then use the second derivative test to determine whether  $TC(q)$  reaches a local maximum, local minimum, or tell me if the test is inconclusive. Clearly put a box around your critical numbers and clearly label each as either local max, local min, or test inconclusive.

**STEP 1**  $TC'(q) = \frac{1}{4}q^2 - q + \frac{3}{4} \stackrel{?}{=} 0 \Rightarrow q^2 - 4q + 3 = 0$   
 $(q-1)(q-3) = 0$  on USE QUAD. FORMULA.

**STEP 2**  $TC''(q) = \frac{1}{2}q - 1$   
 At  $q=1$ ,  $TC''(1) = \frac{1}{2}(1) - 1 = -\frac{1}{2} < 0$   $\wedge$  LOCAL MAX  
 At  $q=3$ ,  $TC''(3) = \frac{1}{2}(3) - 1 = \frac{1}{2} > 0$   $\cup$  LOCAL MIN

$q=1$  LOCAL MAX  
 $q=3$  LOCAL MIN